Miscellaneous Examples

Example 9 Find the value of $\sin^{-1}(\sin\frac{3\pi}{5})$

Solution We know that $\sin^{-1}(\sin x) = x$. Therefore, $\sin^{-1}(\sin\frac{3\pi}{5}) = \frac{3\pi}{5}$

But $\frac{3\pi}{5} \notin \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$, which is the principal branch of $\sin^{-1} x$

However $\sin(\frac{3\pi}{5}) = \sin(\pi - \frac{3\pi}{5}) = \sin\frac{2\pi}{5}$ and $\frac{2\pi}{5} \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$

Therefore $\sin^{-1}(\sin\frac{3\pi}{5}) = \sin^{-1}(\sin\frac{2\pi}{5}) = \frac{2\pi}{5}$

Example 10 Show that $\sin^{-1}\frac{3}{5} - \sin^{-1}\frac{8}{17} = \cos^{-1}\frac{84}{85}$

Solution Let $\sin^{-1}\frac{3}{5} = x$ and $\sin^{-1}\frac{8}{17} = y$

Therefore $\sin x = \frac{3}{5}$ and $\sin y = \frac{8}{17}$

Now $\cos x = \sqrt{1-\sin^2 x} = \sqrt{1-\frac{9}{25}} = \frac{4}{5}$ (Why?)

and $\cos y = \sqrt{1-\sin^2 y} = \sqrt{1-\frac{64}{289}} = \frac{15}{17}$

We have $\cos(x-y) = \cos x \cos y + \sin x \sin y$

$$= \frac{4}{5}\times\frac{15}{17} + \frac{3}{5}\times\frac{8}{17} = \frac{84}{85}$$

Therefore $x - y = \cos^{-1}\left(\frac{84}{85}\right)$

Hence $\sin^{-1}\frac{3}{5} - \sin^{-1}\frac{8}{17} = \cos^{-1}\frac{84}{85}$

Example 11 Show that $\sin^{-1}\frac{12}{13}+\cos^{-1}\frac{4}{5}+\tan^{-1}\frac{63}{16}=\pi$

Solution Let $\sin^{-1}\frac{12}{13}=x,\quad \cos^{-1}\frac{4}{5}=y,\ \tan^{-1}\frac{63}{16}=z$

Then $\quad \sin x=\frac{12}{13},\quad \cos y=\frac{4}{5},\quad \tan z=\frac{63}{16}$

Therefore $\quad \cos x=\frac{5}{13},\ \sin y=\frac{3}{5},\ \tan x=\frac{12}{5}$ and $\tan y=\frac{3}{4}$

We have $\quad \tan(x+y)=\dfrac{\tan x+\tan y}{1-\tan x\tan y}=\dfrac{\frac{12}{5}+\frac{3}{4}}{1-\frac{12}{5}\times\frac{3}{4}}=-\frac{63}{16}$

Hence $\quad \tan(x+y)=-\tan z$

i.e., $\quad \tan(x+y)=\tan(-z)$ or $\tan(x+y)=\tan(\pi-z)$

Therefore $\quad x+y=-z$ or $x+y=\pi-z$

Since $\quad x, y$ and z are positive, $x+y\neq -z$ (Why?)

Hence $\quad x+y+z=\pi$ or $\sin^{-1}\frac{12}{13}+\cos^{-1}\frac{4}{5}+\tan^{-1}\frac{63}{16}=\pi$

Example 12 Simplify $\tan^{-1}\left[\dfrac{a\cos x-b\sin x}{b\cos x+a\sin x}\right]$, if $\frac{a}{b}\tan x>-1$

Solution We have,

$$\tan^{-1}\left[\frac{a\cos x-b\sin x}{b\cos x+a\sin x}\right]=\tan^{-1}\left[\frac{\frac{a\cos x-b\sin x}{b\cos x}}{\frac{b\cos x+a\sin x}{b\cos x}}\right]=\tan^{-1}\left[\frac{\frac{a}{b}-\tan x}{1+\frac{a}{b}\tan x}\right]$$

$$=\tan^{-1}\frac{a}{b}-\tan^{-1}(\tan x)=\tan^{-1}\frac{a}{b}-x$$

Example 13 Solve $\tan^{-1} 2x + \tan^{-1} 3x = \frac{\pi}{4}$

Solution We have $\tan^{-1} 2x + \tan^{-1} 3x = \frac{\pi}{4}$

or $$\tan^{-1}\left(\frac{2x+3x}{1-2x\times 3x}\right) = \frac{\pi}{4}$$

i.e. $$\tan^{-1}\left(\frac{5x}{1-6x^2}\right) = \frac{\pi}{4}$$

Therefore $$\frac{5x}{1-6x^2} = \tan\frac{\pi}{4} = 1$$

or $$6x^2 + 5x - 1 = 0 \text{ i.e., } (6x - 1)(x + 1) = 0$$

which gives $$x = \frac{1}{6} \text{ or } x = -1.$$

Since $x = -1$ does not satisfy the equation, as the L.H.S. of the equation becomes negative, $x = \frac{1}{6}$ is the only solution of the given equation.

- The domains and ranges (principal value branches) of inverse trigonometric functions are given in the following table:

Functions	**Domain**	**Range (Principal Value Branches)**
$y = \sin^{-1} x$	$[-1, 1]$	$\left[\frac{-\pi}{2}, \frac{\pi}{2}\right]$
$y = \cos^{-1} x$	$[-1, 1]$	$[0, \pi]$
$y = \text{cosec}^{-1} x$	$\mathbf{R} - (-1,1)$	$\left[\frac{-\pi}{2}, \frac{\pi}{2}\right] - \{0\}$
$y = \sec^{-1} x$	$\mathbf{R} - (-1, 1)$	$[0, \pi] - \{\frac{\pi}{2}\}$
$y = \tan^{-1} x$	$\mathbf{R}$	$\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$
$y = \cot^{-1} x$	$\mathbf{R}$	$(0, \pi)$

- $\sin^{-1}x$ should not be confused with $(\sin x)^{-1}$. In fact $(\sin x)^{-1} = \frac{1}{\sin x}$ and similarly for other trigonometric functions.
- The value of an inverse trigonometric functions which lies in its principal value branch is called the *principal value* of that inverse trigonometric functions.

For suitable values of domain, we have

- $y = \sin^{-1} x \Rightarrow x = \sin y$ $\quad$ $x = \sin y \Rightarrow y = \sin^{-1} x$
- $\sin(\sin^{-1} x) = x$ $\quad$ $\sin^{-1}(\sin x) = x$
- $\sin^{-1} \frac{1}{x} = \text{cosec}^{-1} x$ $\quad$ $\cos^{-1}(-x) = \pi - \cos^{-1} x$
- $\cos^{-1} \frac{1}{x} = \sec^{-1} x$ $\quad$ $\cot^{-1}(-x) = \pi - \cot^{-1} x$
- $\tan^{-1} \frac{1}{x} = \cot^{-1} x$ $\quad$ $\sec^{-1}(-x) = \pi - \sec^{-1} x$

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$$\sin^{-1}(-x) = -\sin^{-1} x$$

$$\tan^{-1} x + \cot^{-1} x = \frac{\pi}{2}$$

$$\sin^{-1} x + \cos^{-1} x = \frac{\pi}{2}$$

$$\tan^{-1}x + \tan^{-1}y = \tan^{-1} \frac{x+y}{1-xy}$$

$$\tan^{-1}x - \tan^{-1}y = \tan^{-1} \frac{x-y}{1+xy}$$

$$2\tan^{-1} x = \sin^{-1} \frac{2x}{1+x^2} = \cos^{-1} \frac{1-x^2}{1+x^2}$$

$$\tan^{-1}(-x) = -\tan^{-1} x$$

$$\operatorname{cosec}^{-1}(-x) = -\operatorname{cosec}^{-1} x$$

$$\operatorname{cosec}^{-1} x + \sec^{-1} x = \frac{\pi}{2}$$

$$2\tan^{-1}x = \tan^{-1} \frac{2x}{1-x^2}$$

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